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# Warped product submanifolds in Sasakian space forms

**Koji Matsumoto and Ion Mihai***

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**Abstract.** Recently, Chen established a general sharp inequality for warped products in real space forms. As applications, he obtained obstructions to minimal isometric immersions of warped products into real space forms.
In the present paper, we obtain sharp inequalities for warped products isometrically immersed in Sasakian space forms. Some applications are derived.

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## §1. Introduction

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and $f$ a positive differentiable function on $M_1$. The warped product of $M_1$ and $M_2$ is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [4]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3].

Let $x : M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\widetilde{M}(c)$ with constant sectional curvature $c$. We denote by $h$ the second fundamental form of $x$ and $H_i = \frac{1}{n_i} \operatorname{trace} h_i$,

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where trace $h_i$ is the trace of $h$ restricted to $M_i$ and $n_i = \dim M_i$ $(i = 1, 2)$. We call $H_i$ $(i = 1, 2)$ the partial mean curvature vectors.

The immersion $x$ is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields $X$ and $Z$ tangent to $M_1$ and $M_2$ respectively.

In [4], Chen established the following sharp relationship between the warping function $f$ of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

**Theorem 1.1.** *Let $x$ be an isometric immersion of an $n$-dimensional warped product $M_1 \times_f M_2$ into an $m$-dimensional Riemannian manifold $\widetilde{M}(c)$ of constant holomorphic sectional curvature $c$. Then*:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 c, \tag{1.1}$$

*where $n_i = \dim M_i, i = 1, 2$, and $\Delta$ is the Laplacian operator of $M_1$.*

*Moreover, the equality case of* (1.1) *holds if and only if $x$ is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.*

As applications, the author obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form (see [4]). Examples of submanifolds satisfying the equality case of (1.1) are given.

## §2. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism $\phi$ of its tangent bundle $T\widetilde{M}$, a vector field $\xi$ and a 1-form $\eta$, satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \ \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields $X, Y$ on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to $g$.

A plane section $\pi$ in $T_p\tilde{M}$ is called a $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature of a $\phi$-section is called a $\phi$-sectional curvature. A Sasakian manifold with constant $\phi$-sectional curvature $c$ is said to be a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor of $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by [1]

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}+ \tag{2.1}$$

$$+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi+$$

$$+g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\},$$

for any tangent vector fields $X, Y, Z$ on $\widetilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and $S^{2m+1}$, with standard Sasakian structures (see [1], [8]).

Let $M$ be an $n$-dimensional submanifold in a Sasakian space form $\widetilde{M}(c)$ of constant $\phi$-sectional curvature $c$. We denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM, p \in M$, and $\nabla$ the Riemannian connection of $M$, respectively. Also, let $h$ be the second fundamental form and $R$ the Riemann curvature tensor of $M$.

Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W)+ \tag{2.2}$$

$$+g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

for any vectors $X, Y, Z, W$ tangent to $M$.

Let $p \in M$ and $\{e_1, ..., e_n, ..., e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}(c)$, such that $e_1, ..., e_n$ are tangent to $M$ at $p$. We denote by $H$ the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{2.3}$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, ..., n\}, r \in \{n+1, ..., 2m+1\}. \tag{2.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.5}$$

For any tangent vector field $X$ to $M$, we put $\phi X = PX + FX$, where $PX$ and $FX$ are the tangential and normal components of $\phi X$, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{2.6}$$

We recall the following result of Chen for later use.

**Lemma [2].** *Let $n \geq 2$ and $a_1, ..., a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right)$$

*Then $2a_1a_2 \geq b$, with equality holding if and only if*

$$a_1 + a_2 = a_3 = ... = a_n.$$

## §3. $C$-totally real warped product submanifolds

Chen established a sharp relationship between the warping function $f$ of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [4]). We prove similar inequalities for warped product submanifolds of a Sasakian space form.

In this section, we investigate $C$-totally real warped product submanifolds in a Sasakian space form $\widetilde{M}(c)$.

A submanifold $M$ normal to $\xi$ in a Sasakian space form $\widetilde{M}(c)$ is said to be a *$C$-totally real submanifold*. It follows that $\phi$ maps any tangent space of $M$ into the normal space, that is $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

**Theorem 3.1.** *Let $x$ be a $C$-totally real isometric immersion of an $n$-dimensional warped product $M_1 \times_f M_2$ into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then*:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4}, \tag{3.1}$$

*where $n_i = \dim M_i, i = 1, 2,$ and $\Delta$ is the Laplacian operator of $M_1$.*

*Moreover, the equality case of (3.1) holds if and only if $x$ is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where $H_i, i = 1, 2,$ are the partial mean curvature vectors.*

*Proof.* Let $M_1 \times_f M_2$ be a $C$-totally real warped product submanifold into a Sasakian space form $\widetilde{M}(c)$ of constant $\phi$-sectional curvature $c$.

Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{3.2}$$

for any vector fields $X, Z$ tangent to $M_1, M_2$, respectively.

If $X$ and $Z$ are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by $X$ and $Z$ is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{3.3}$$

We choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$, such that $e_1, ..., e_{n_1}$ are tangent to $M_1$, $e_{n_1+1}, ..., e_n$ are tangent to $M_2$, $e_{n+1}$ is parallel to the mean curvature vector $H$ and $e_{2m+1} = \xi$.

Then, using (3.3), we get

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{3.4}$$

for each $s \in \{n_1 + 1, ..., n\}$.

From the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3}{4}, \tag{3.5}$$

where $\tau$ denotes the scalar curvature of $M_1 \times_f M_2$, that is,

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We set

$$\delta = 2\tau - n(n-1)\frac{c+3}{4} - \frac{n^2}{2}\|H\|^2. \tag{3.6}$$

Then, (3.5) can be written as

$$n^2 \|H\|^2 = 2(\delta + \|h\|^2). \tag{3.7}$$

With respect to the above orthonormal frame, (3.7) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i \ne j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1 \le i \ne j \le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \right.$$

$$- \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1} h_{tt}^{n+1} \Bigg\}.$$

Thus $a_1, a_2, a_3$ satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right),$$

with

$$b = \delta + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1} h_{tt}^{n+1}.$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1} h_{tt}^{n+1} \ge \tag{3.8}$$

$$\ge \frac{\delta}{2} + \sum_{1\le \alpha<\beta\le n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^{r})^2.$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.9}$$

Using again the Gauss equation, we have

$$n_2 \frac{\Delta f}{f} = \tau - \sum_{1\le j<k\le n_1} K(e_j \wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s \wedge e_t) = \tag{3.10}$$

$$= \tau - \frac{n_1(n_1-1)(c+3)}{8} - \sum_{r=n+1}^{2m} \sum_{1\le j<k\le n_1} (h_{jj}^{r} h_{kk}^{r} - (h_{jk}^{r})^2) -$$

$$- \frac{n_2(n_2-1)(c+3)}{8} - \sum_{r=n+1}^{2m} \sum_{n_1+1\le s<t\le n} (h_{ss}^{r} h_{tt}^{r} - (h_{st}^{r})^2).$$

Combining (3.8) and (3.10) and taking account of (3.4), we obtain

$$n_2 \frac{\Delta f}{f} \le \tau - \frac{n(n-1)(c+3)}{8} + n_1 n_2 \frac{c+3}{4} - \frac{\delta}{2} - \tag{3.11}$$

$$-\sum_{1\le j\le n_1;n_1+1\le t\le n}(h_{jt}^{n+1})^2-\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^r)^2+$$

$$+\sum_{r=n+2}^{2m}\sum_{1\le j<k\le n_1}((h_{jk}^r)^2-h_{jj}^rh_{kk}^r)+\sum_{r=n+2}^{2m}\sum_{n_1+1\le s<t\le n}((h_{st}^r)^2-h_{ss}^rh_{tt}^r)=$$

$$=\tau-\frac{n(n-1)(c+3)}{8}+n_1n_2\frac{c+3}{4}-\frac{\delta}{2}-\sum_{r=n+1}^{2m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(h_{jt}^r)^2-$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{j=1}^{n_1}h_{jj}^r\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{t=n_1+1}^{n}h_{tt}^r\right)^2\le$$

$$\le\tau-\frac{n(n-1)(c+3)}{8}+n_1n_2\frac{c+3}{4}-\frac{\delta}{2}=$$

$$=\frac{n^2}{4}\|H\|^2+n_1n_2\frac{c+3}{4},$$

which implies the inequality (3.1).

We see that the equality sign of (3.11) holds if and only if

$$h_{jt}^r=0,\quad 1\le j\le n_1, n_1+1\le t\le n, n+1\le r\le 2m, \tag{3.12}$$

and

$$\sum_{i=1}^{n_1}h_{ii}^r=\sum_{t=n_1+1}^{n}h_{tt}^r=0,\quad n+2\le r\le 2m. \tag{3.13}$$

Obviously (3.12) is equivalent to the mixed totally geodesicness of the warped product $M_1\times_f M_2$ and (3.9) and (3.13) implies $n_1H_1=n_2H_2$.

The converse statement is straightforward. □

As applications, we derive certain obstructions to the existence of minimal $C$-totally real warped product submanifolds in Sasakian space forms.

**Corollary 3.2.** *Let $M_1\times_f M_2$ be a warped product whose warping function $f$ is harmonic. Then*:

(i) *$M_1\times_f M_2$ admits no minimal $C$-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c<-3$.*

(ii) *Every minimal $C$-totally real immersion of $M_1\times_f M_2$ in the standard Sasakian space form $\mathbb{R}^{2m+1}$ is a warped product immersion.*

*Proof.* Assume $f$ is a harmonic function on $M_1$ and $M_1\times_f M_2$ admits a minimal $C$-totally real immersion in a Sasakian space form $\widetilde{M}(c)$. Then, the inequality (3.1) becomes $c\ge -3$.

If $c = -3$, the equality case of (3.1) holds. By Theorem 3.1, it follows that $M_1 \times_f M_2$ is mixed totally geodesic and $H_1 = H_2 = 0$. A well-known result of Nölker [7] implies that the immersion is a warped product immersion. □

**Corollary 3.3.** *If the warping function $f$ of a warped product $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on $M_1$ with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ does not admit a minimal $C$-totally real immersion in a Sasakian space form $\widetilde{M}(c)$ with $c \leq -3$.*

We give an example of a $C$-totally real submanifold which satisfies the equality case of (3.1).

Consider $S^5 \subset S^7$ and let $n$ be a unit vector orthogonal to the linear subspace containing $S^5$. Let $N$ be any minimal $C$-totally real surface of $S^5$ and define the warped product manifold

$$M = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} N.$$

It is isometrically immersed in $S^7$ by

$$\psi : M \to S^7, \quad \psi(t,p) = (\sin t)n + (\cos t)p.$$

This immersion is $C$-totally real and satisfies the equality case of (3.1) (see also [5]).

## §4. Warped product submanifolds tangent to the Reeb vector field $\xi$

In this section, we investigate warped product submanifolds tangent to the structure vector field $\xi$ in a Sasakian space form $\widetilde{M}(c)$.

We distinguish 2 cases:
(a) $\xi$ is tangent to $M_1$;
(b) $\xi$ is tangent to $M_2$.

**Theorem 4.1.** *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and $M_1 \times_f M_2$ an $n$-dimensional warped product submanifold, such that $\xi$ is tangent to $M_1$. Then:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4} - \frac{c-1}{4}, \tag{4.1}$$

*where $n_i = \dim M_i, i = 1, 2$, and $\Delta$ is the Laplacian operator of $M_1$.*

*Moreover, the equality case of (4.1) holds if and only if $M_1 \times_f M_2$ is a mixed totally geodesic submanifold and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.*

*Proof.* Let $M_1 \times_f M_2$ be a warped product submanifold of a Sasakian space form $\widetilde{M}(c)$ with constant $\phi$-sectional curvature $c$, such that $\xi$ is tangent to $M_1$. It is obvious that $M_2$ is a $C$-totally real submanifold of $\widetilde{M}(c)$.

We choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ such that $e_1, ..., e_{n_1} = \xi$ are tangent to $M_1$, $e_{n_1+1}, ..., e_n$ are tangent to $M_2$ and $e_{n+1}$ is parallel to $H$.

From the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n + 2)\frac{c-1}{4}. \tag{4.2}$$

We denote

$$\delta = 2\tau - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n + 2)\frac{c-1}{4} - \frac{n^2}{2}\|H\|^2. \tag{4.3}$$

Then, (4.2) takes the form

$$n^2\|H\|^2 = 2(\delta + \|h\|^2). \tag{4.4}$$

We will use the same method as in the proof of Theorem 3.1. We will point-out only the differences.

Using again the Gauss equation, we obtain

$$n_2\frac{\Delta f}{f} = \tau - \sum_{1\leq j<k\leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1\leq s<t\leq n} K(e_s \wedge e_t) = \tag{4.5}$$

$$= \tau - \frac{n_1(n_1-1)(c+3)}{8} - \left[3\sum_{1\leq j<k\leq n_1-1} g^2(Pe_j, e_k) - n_1 + 1\right]\frac{c-1}{4} -$$

$$-\sum_{r=n+1}^{2m+1}\sum_{1\leq j<k\leq n_1}(h^r_{jj}h^r_{kk} - (h^r_{jk})^2) - \frac{n_2(n_2-1)(c+3)}{8} - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\leq s<t\leq n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2).$$

Applying the Lemma of Chen to (4.4) and substituting (4.5), similar computations as in the proof of Theorem 3.1 lead to

$$n_2\frac{\Delta f}{f} \leq \tau - \frac{n(n-1)(c+3)}{8} + n_1n_2\frac{c+3}{4} - \frac{\delta}{2} - \tag{4.6}$$

$$-\left[3\sum_{1\leq j<k\leq n_1-1} g^2(Pe_j, e_k) - n_1 + 1\right]\frac{c-1}{4}.$$

Using (4.3), the inequality (4.6) becomes

$$n_2\frac{\Delta f}{f} \leq \frac{n^2}{4}\|H\|^2 + n_1n_2\frac{c+3}{4} - n_2\frac{c-1}{4},$$

i.e. the inequality to prove.

The equality case of the inequality (4.1) is similar to the equality case of (3.1). □

Assume now that $M_1 \times_f M_2$ is a warped product submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $\xi$ is tangent to $M_2$.

If we put $Z = \xi$ in (3.2), it follows that $Xf = 0$, for all vector fields $X$ tangent to $M_1$. Thus $f$ is constant and the warped product becomes a Riemannian product.

**Proposition 4.2.** *Any warped product submanifold $M_1 \times_f M_2$ of a Sasakian space form $\widetilde{M}(c)$ such that $\xi$ is tangent to $M_2$ is a Riemannian product.*

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Koji Matsumoto
Department of Mathematics, Faculty of Education, Yamagata University
990-8560 Yamagata, Japan
*E-mail*: `ej192@kdw.kj.yamagata-u.ac.jp`

Ion Mihai
Faculty of Mathematics, University of Bucharest
Str. Academiei 14, 70109 Bucharest, Romania
*E-mail*: `imihai@math.math.unibuc.ro`